Multitiered experiments: Tiers in the design and analysis of experiments

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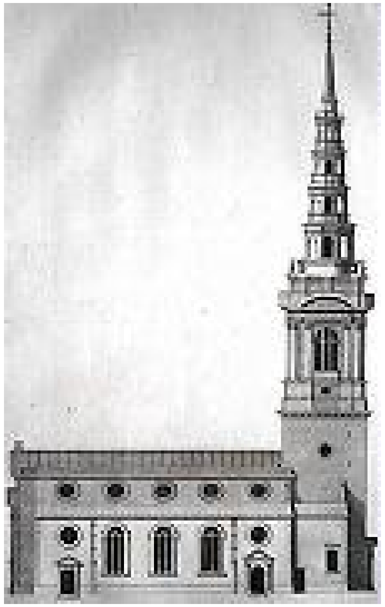
Multitiered experiments course outline

I. Introduction and two-tiered experiments
II. Multiple randomizations
III. Randomization-based analysis of experiments

I. Introduction & two-tiered experiments

1. Multitiered experiments: what are they?
2. Two-tiered experiments
   i. Notation
   ii. The design question
3. Summary

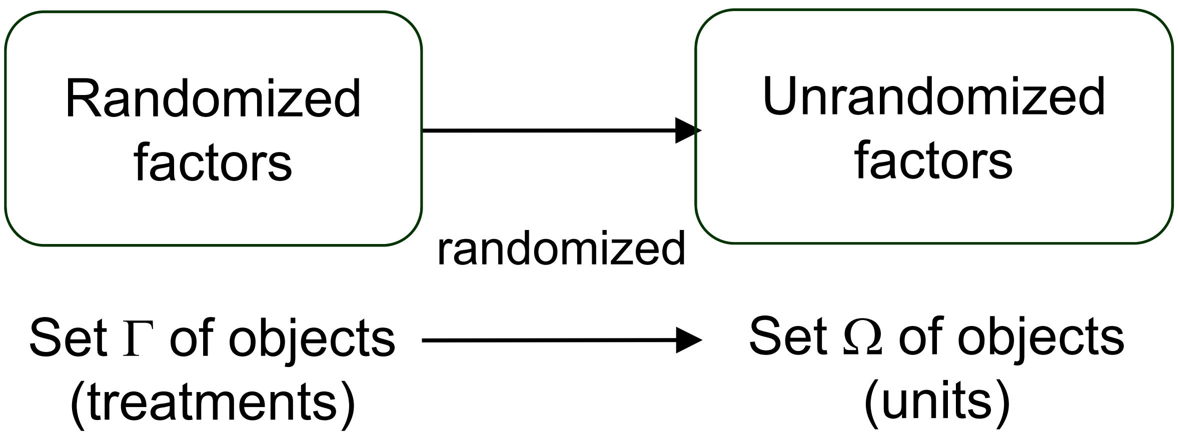


1. Multitiered experiments: what are they?

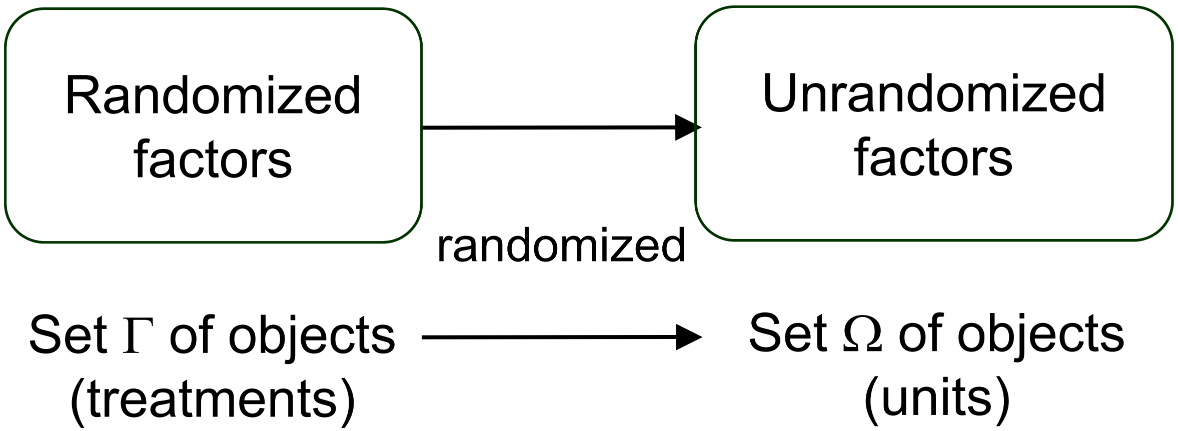
- Experiments involving multiple randomizations
- Include:
  - two-phase experiments (sensory experiments in 1970s but McIntyre, 1955);
  - superimposed experiments;
  - some plant and animal experiments;
  - human interaction experiments;
  - multistage reprocessing experiments.
- Not new designs but a particular class of (complex) experiments.
- How many randomizations in a design:
  - One in an RCBD?
  - Two in a split-plot design?
  - Two in a two-phase experiment?
- Depends on how randomization is defined

A definition of a randomization

- Define a randomization to be the random assignment of one set of objects to another using a permutation of the latter.
- Generally each set of objects is indexed by a set of factors
  - Unrandomized factors (indexing units)
  - Randomized factors (indexing treatments)
- A permutation of units used to randomize



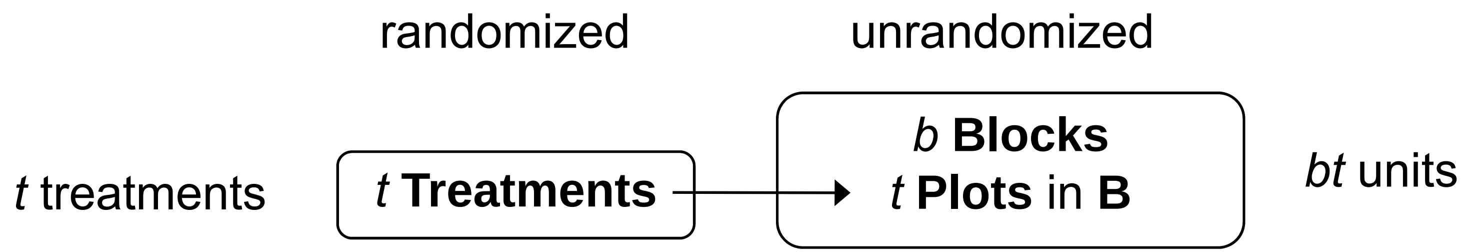
How is such a randomization achieved?



1. Write down an unrandomized list of a) the set Ω; b) the levels of the unrandomized factors in standard order; c) the randomized factors in systematic order according to the design being used;
2. Identify all possible permutations of the levels combinations of factors indexing Ω allowable for the design;
3. Select a permutation and apply it to the levels combinations of factors indexing Ω.

   Now have levels combinations of all factors that will occur in experiment.

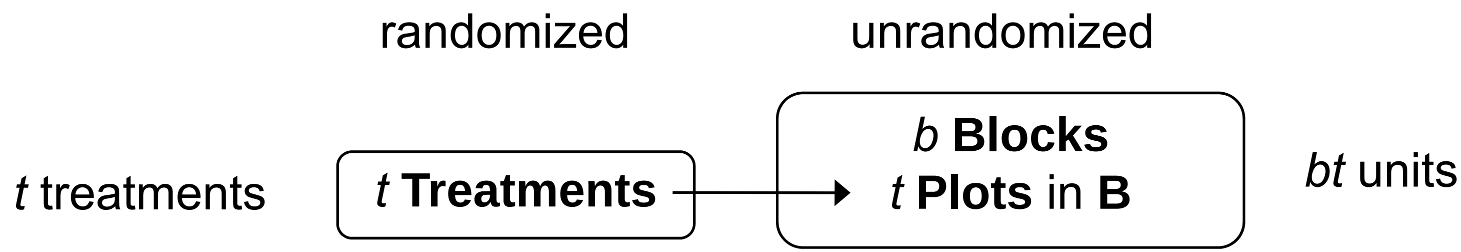
A randomization



- Systematic design:
  - one treatment on each plot in each block
- Randomization:
  - permute blocks
  - permute plots in each block independently

Unit	Blocks	Plots	Treatments
1	1	1	1
2	1	2	2
3	2	1	1
4	2	2	2

A randomization



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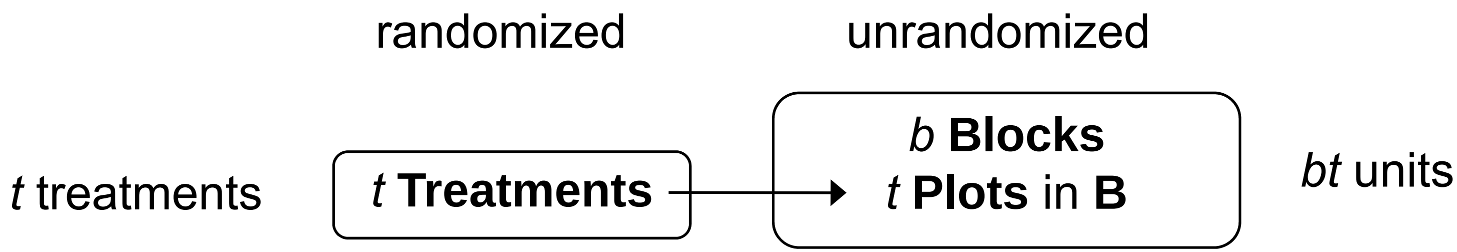

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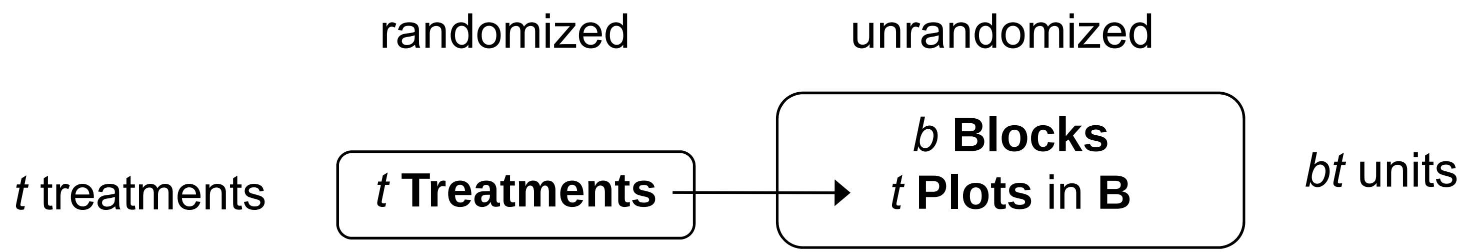
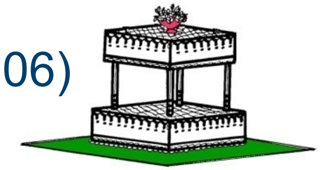


- Systematic design:
  - one treatment on each plot in each block

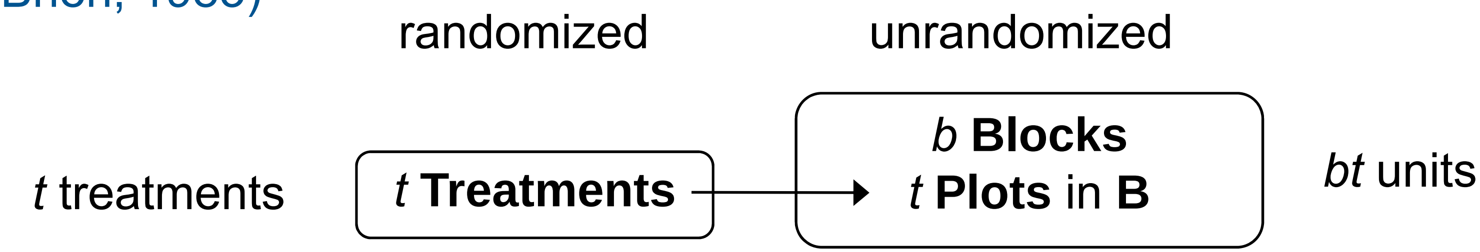
- Randomization:
  - permute blocks
  - permute plots in each block independently

- The arrow from the randomized tier to the unrandomized tier indicates both
  a) a systematic design (with extra explanation if necessary)
  b) the randomization: permute the (names of the) objects in the unrandomized set by a permutation chosen at random from among all those that preserve the relevant structure.

Randomization diagrams (Brien & Bailey, 2006)



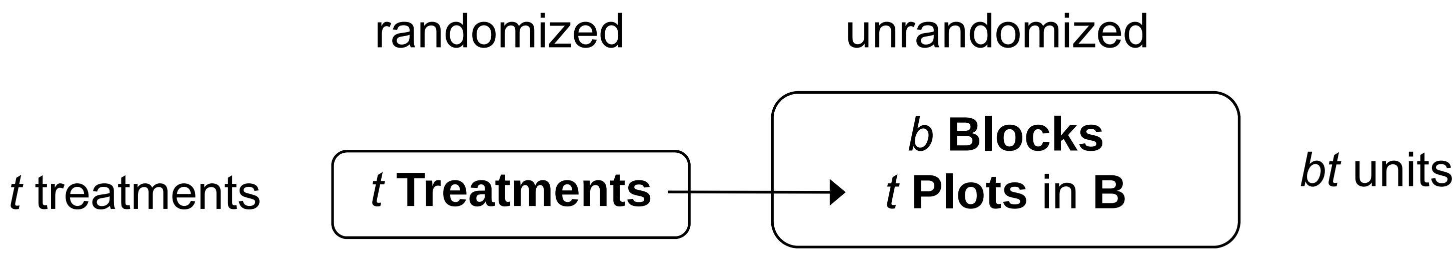
Example 1: RCBD in first phase of a sensory experiment
(Brien, 1983)



- A **panel** for a set of objects shows a factor poset:
  - a list of factors in a tier,
  - their numbers of levels,
  - their nesting relationships.
- So a **tier** is just a set of factors:
  - {Treatments} or {Blocks, Plots}
  - But, not just any old set: a set of factors with the same status in the randomization.
  - Factors in different tiers have the levels combinations of them that are observed determined by a randomization.
  - Not true for any factors in the same tier.
- Shows EU and restrictions placed on randomization.

Standard textbook examples

RCBD

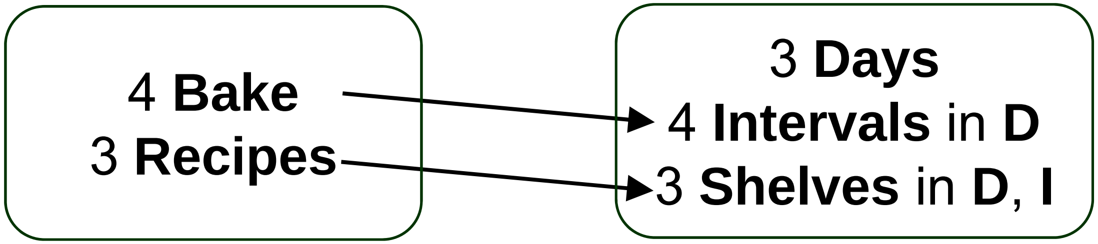


- Most textbook examples involve two tiers only and tier is unnecessary.
- Could have factor sets:
  - Topographical and treatment (Fisher, 1935)
  - Block and treatment (Nelder, 1965a, b)
  - Plots/Units and treatments (Wilks & Kempthorne, 1956; Bailey, 1981)
  - Unrandomized and randomized (Brien, 1983)

Randomization for a standard split-plot experiment

An experiment about cooking cakes in an oven

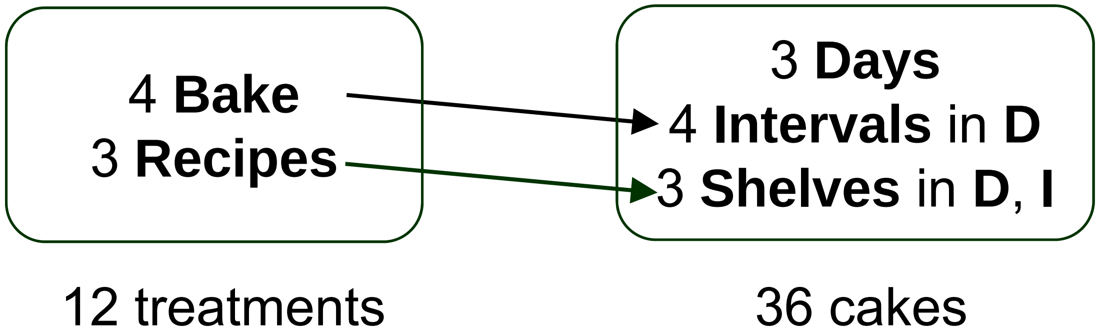
- *Main plots*
  - 4 Baking conditions randomized to 4 Intervals in each of 3 Days using RCBD.
- *Subplots*
  - In an Interval, an oven with 3 Shelves used to bake 3 different Recipes together.
  - Recipes order on Shelves randomized in each Interval.



12 treatments

36 cakes

Sense in which one randomization

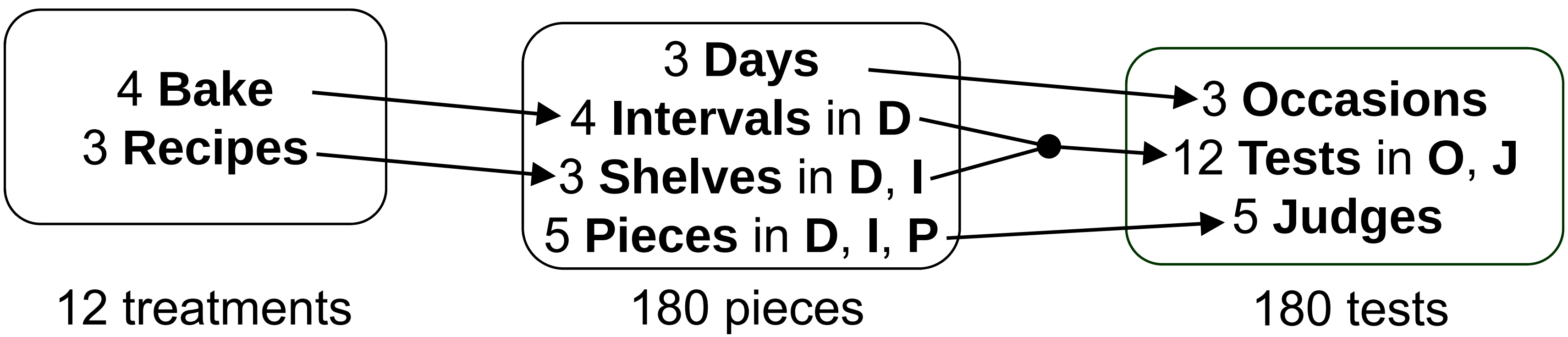


- Two randomizations in that two factors are randomized to two experimental units:
  - Bake to Intervals in Days
  - Recipes to Shelves in Days and Intervals
- But, as randomization has been defined, one randomization:
  - can achieve by applying one of the allowable permutations of *cakes*.
  - Two set of objects with tiers

    factors associated by randomization in different tiers.
- Presume cakes measured (physical?) immediately:
  - A complete experiment

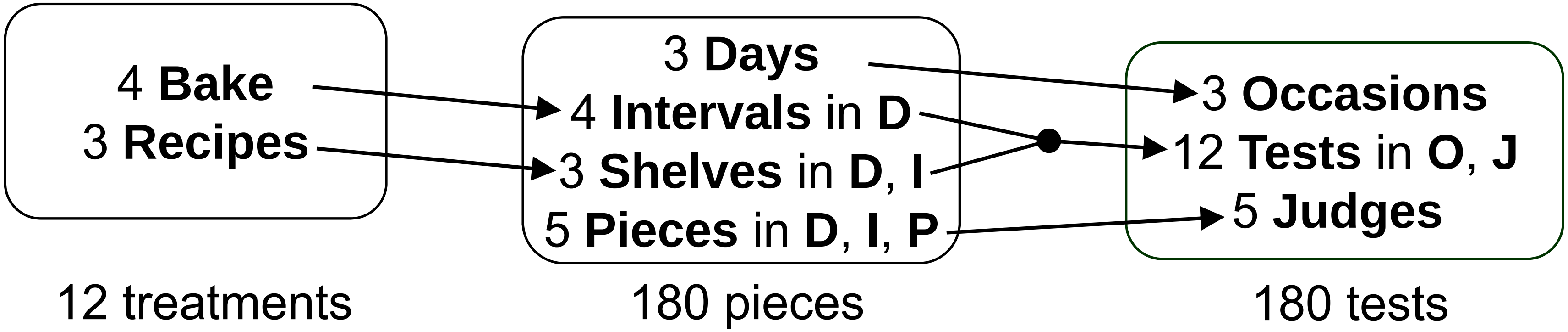
Extend to second phase

- Suppose 36 cakes to be evaluated in a sensory phase.
- Sensory phase:
  - 5 Judges are to be given a piece of cake to assess.
  - Sensory phase divided into 3 Occasions at which 12 cakes, all produced on the same day, are assessed one at a time by the 5 judges.
  - Order of presentation randomized for each judge on each occasion.



- Three sets of objects
- Two randomizations whose effects cannot be achieved with one

Fundamental difference between split-plot and two-phase experiments



- Split-plots involve **independent randomizations** (Koch, Elashoff & Amara, 1988, Encyclopaedia of Statistics) to two different experimental units from the same set of objects.
  - reducible to a single randomization (permutation)
- Two-phase experiments involve two randomizations each to a different set of objects.
  - not reducible to a single randomization.
- Sets of objects and tiers basic to identifying multiple randomizations.

2. Two-tiered experiments

- Tiers can be useful in
  - the design of experiments
  - the analysis of experiments

  especially multitiered experiments
- Establish concepts in using tiers in familiar context of textbook experiments
  - one way of approaching design and analysis of experiments
  - same as Nelder (1965a, b) for two-tiered experiments

2(i) Notation

Factor relationship operators:

A*B factors A and B are crossed

A/B factor B is nested within A

Generalized factor

$A \wedge B$ is the *ab*-level factor formed from the combinations of A with *a* levels and B with *b* levels

Sources in Decomposition/ANOVA table

A#B a source for the interaction of A and B

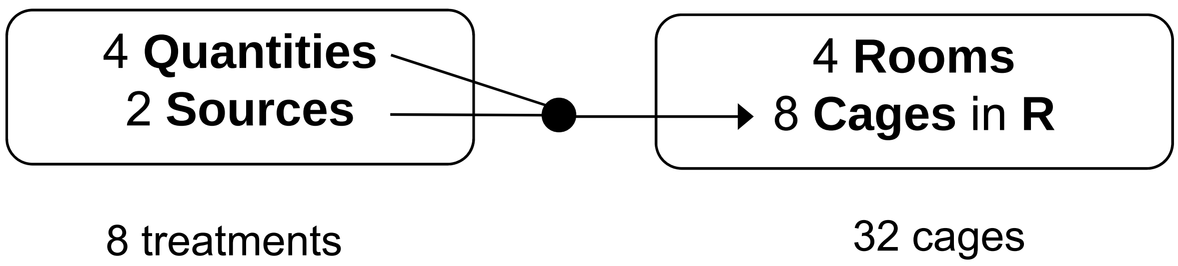
B[A] a source for the effects of B nested within A

$$A / B / C = A + A \wedge B + A \wedge B \wedge C \Rightarrow A + B[A] + C[A \wedge B]$$

Rule: factors, and only those factors, that nest any of the factors in the generalized factor must be in the square braces

A poultry-feeding experiment

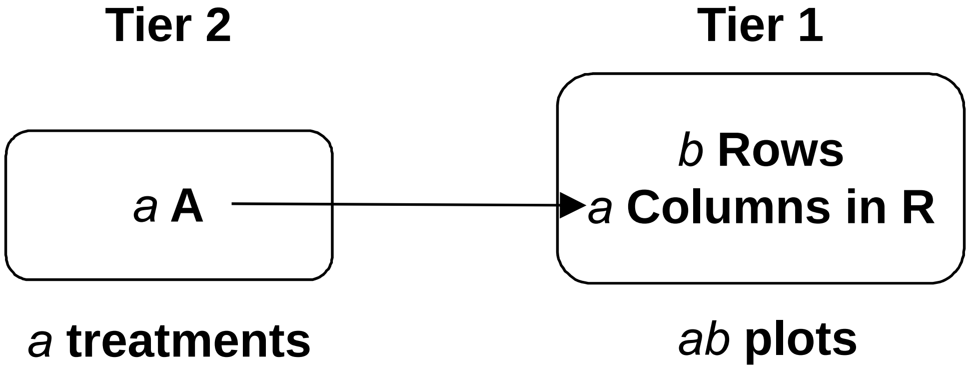
(Brien & Bailey, 2006, Example 2)



- The generalized factor Q∧S has 8 levels.
- R∧C = **generalized factor** whose levels are the 32 combinations of the levels of Rooms and Cages.
- “Cages is **nested** in R” means that R∧C is a meaningful factor but C is not.

2(ii) The design question using tiers

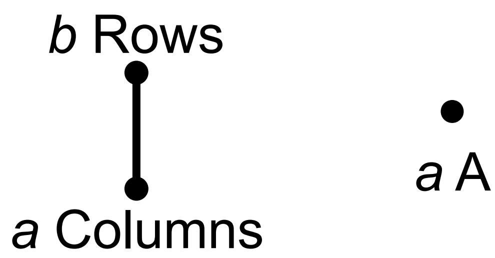
- Two sets of **objects**, one to be randomized to the other.
- For example:
  - A set of plots to which a set of treatments are to be randomized;
  - i.e. objects are plots and treatments.
- Suppose
  - plots in a rectangular array indexed by Rows and Columns;
  - treatments indexed by a single factor A;
  - treatments are assigned to plots using an RCBD with Rows as the blocks.

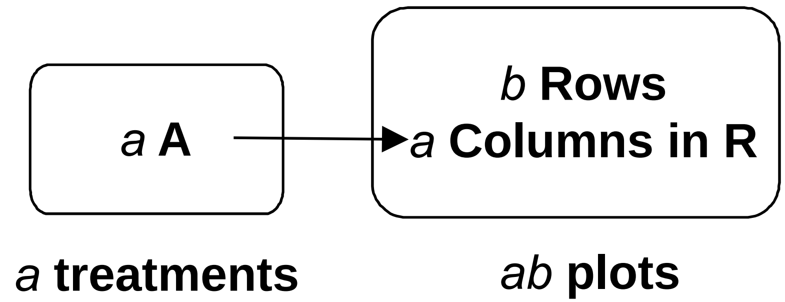


- What are the properties of the design?

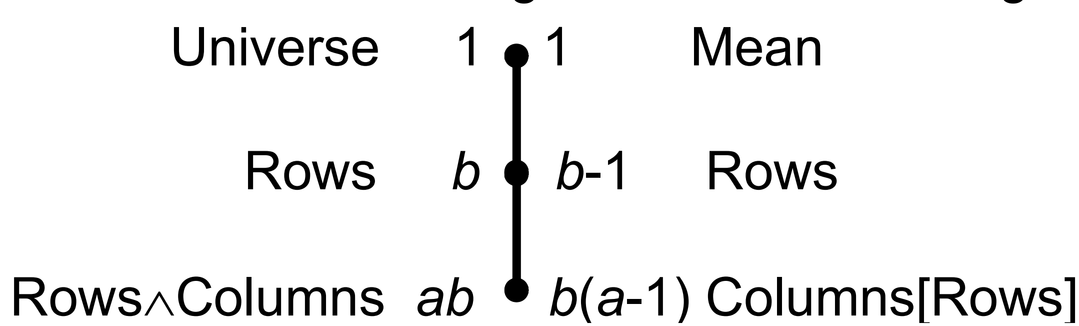
Deriving the sources

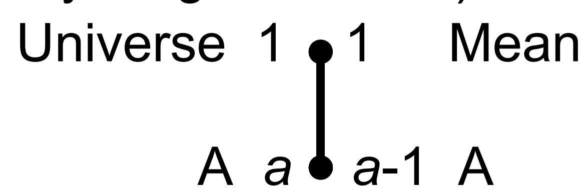
- Objects: plots and treatments
- Tiers: $\mathcal{F}_{\text{plots}}$ = {Rows, Columns}, $\mathcal{F}_{\text{treatments}}$ = {A}
- Factor posets (based on nesting for design and intrinsic):





- Structure formulae
  - *b* Rows / *a* Columns
  - *a* A
- Generalized factors and Sources
  (expand formulae or derive factor combinations complying with nesting & construct Hasse diagrams based on marginality of gen. factors)





Decomposition (skeleton ANOVA) table

- Summarizes properties of design:
  - confounding and apportionment of variability
  - useful at design stage irrespective of whether analysis by ANOVA or mixed model estimation (more later)
- Have from each panel:

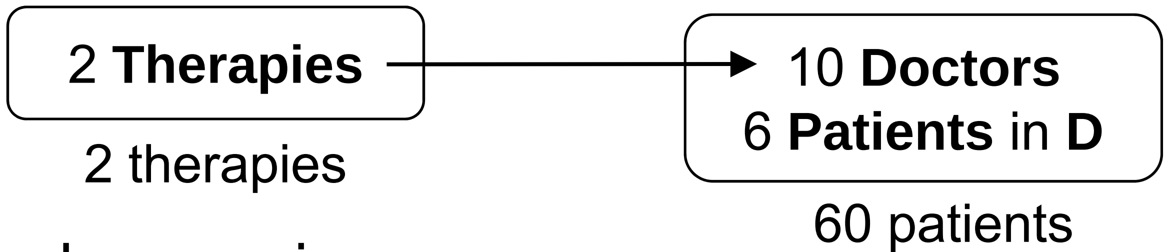
plots tier		treatments tier	
source	df	source	df
Rows	$b - 1$		
Columns[Rows]	$b(a - 1)$	A	$a - 1$
		Residual	$(a - 1)(b - 1)$

treatments tier	
source	df
A	$a - 1$

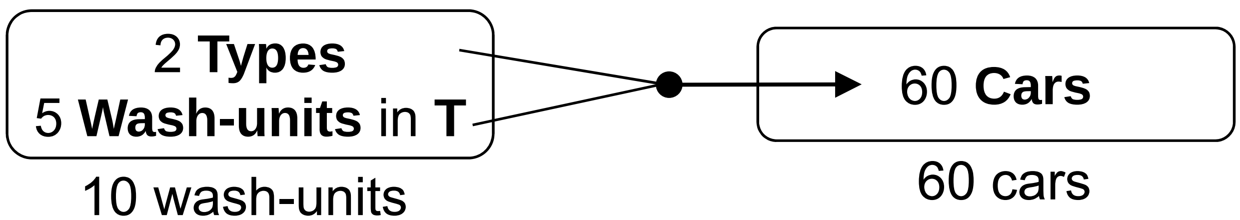
- This table
  - exhibits confounding resulting from randomizations, with multiple sources for contrasts: A is confounded with Columns[Rows]
  - shows us that Columns[Rows], but not Rows, variability affects A differences and that the df of the relevant Residual is $(a - 1)(b - 1)$ df.
- Why is this a nice design?
  - Because it is orthogonal with Rows differences eliminated.

White's (1975) three experiments

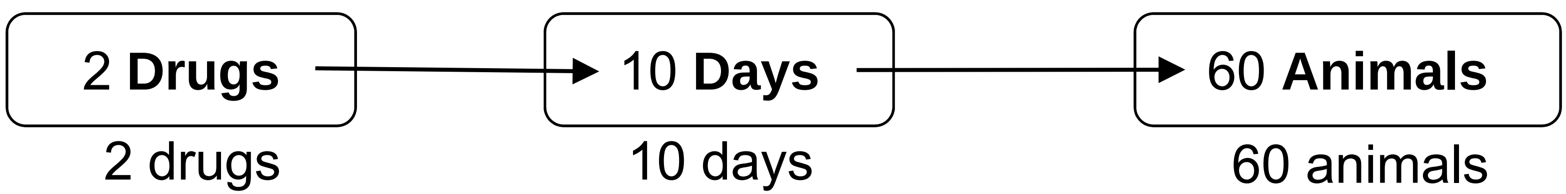
- A cluster-randomized clinical trial



- A car-wash emporium



- An animal experiment



- What are the tiers?
- All could be analyzed using a nested ANOVA (2 **A** / 5 **B** / 6 **C**), but different confounding (or different EUs).
  - different decomposition tables e.g clinical trial

The properties of a design

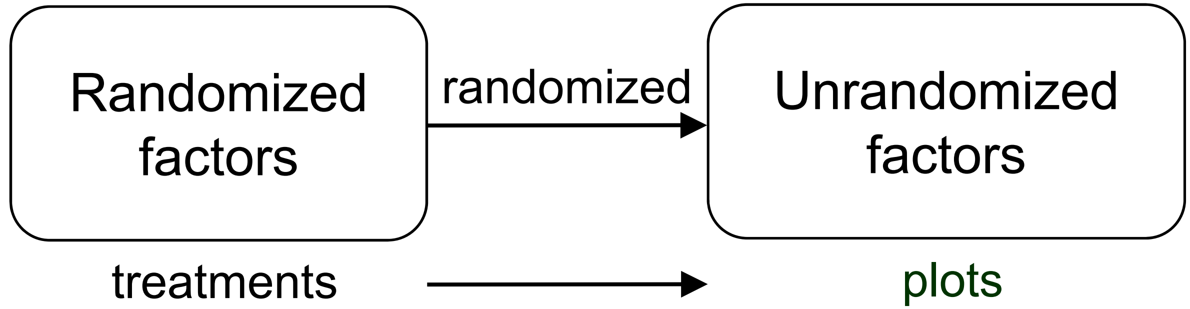
- How do we determine the properties of design in general?

- What happens if not orthogonal?

- Nelder, James-Wilkinson, Brien-Bailey answer by examining the relationship between the **structures** (**orthogonal decompositions)** associated with the two sets of objects.

- The factor poset, specifying the nesting (and crossing) relations between factors in a set, for each tier gives rise to a Poset structure:
  - The orthogonal decomposition for the set all generalized factors that comply with the nesting of the factors in the factor poset. It is specified by a set of orthogonal projectors.

A randomization: its elements



Sets of objects	Γ (treatments)	Ω (plots)
Tiers: sets of factors	$\mathcal{F}_\Gamma$ (Randomized)	$\mathcal{F}_\Omega$ (Unrandomized)
Vector spaces	$V_\Gamma \leq V_\Omega$	$V_\Omega = R^n$
Orthogonal decompositions (structures)	$\mathcal{R}$, a set of **R**s	$\mathcal{U}$, a set of **U**s

Start with $\mathcal{U}$ and further decompose according to $\mathcal{R}$.

An orthogonal decomposition

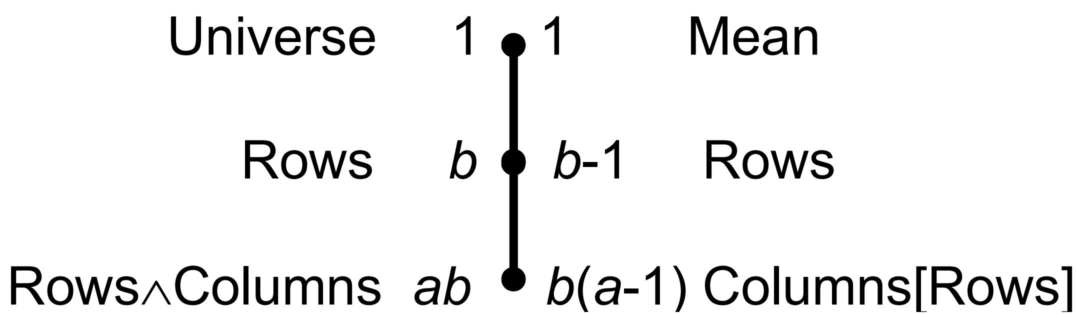
- $\mathcal{U}$ is a set of orthogonal idempotents, $\mathbf{U}_1,\ldots,\mathbf{U}_u$, that decomposes V_Ω into a set of mutually orthogonal subspaces. The idempotents are:
  - Symmetric: $\mathbf{U}' = \mathbf{U}$
  - Idempotent: $\mathbf{U}^2 = \mathbf{U}$
  - Mutually orthogonal: $\mathbf{U}_i\mathbf{U}_j = \mathbf{0}$, $i \neq j$
  - Sum to **I**: $\sum \mathbf{U} = \mathbf{I}$
- Similarly $\mathcal{R}$ is a set of orthogonal idempotents, $\mathbf{R}_1,\ldots,\mathbf{R}_r$, that decomposes V_Γ into a set of mutually orthogonal subspaces, except:
  - Sum to $\mathbf{I}_\mathcal{R}$, the identity for V_Γ in V_Ω (i.e. $\mathbf{R}\mathbf{I}_\mathcal{R} = \mathbf{I}_\mathcal{R}\mathbf{R} = \mathbf{R}$)

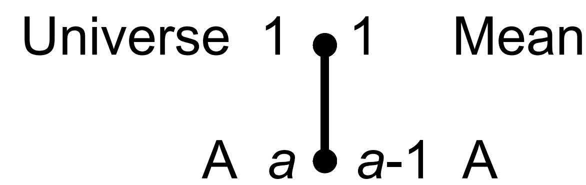
How are $\mathcal{U}$ and $\mathcal{R}$ determined?

- Permutations that conform to the relations for the factor poset mathematically determine the idempotents.
- Alternatively can think of these idempotents as the orthogonal projectors on the orthogonal subspaces corresponding to the terms in a randomization-equivalent mixed model (for a randomization involving two factor posets)
  - The form of the mixed model is $\mathbf{Y} = \mathbf{X}\theta + \mathbf{Z}\mathbf{u}$ with
    - all random terms ($\mathbf{Z}_j\mathbf{u}_j$) from unrandomized, generalized factors
    - all fixed terms ($\mathbf{X}_i\theta_i$) from just randomized, generalized factors.
  - Need the idempotents that project onto the orthogonal subspaces corresponding to fitting either the fixed or random terms.
  - Define summation or relationship matrices $\mathbf{S} = \mathbf{X}\mathbf{X}'$ (or $\mathbf{Z}\mathbf{Z}'$) .
  - Those for a tier determine a relationship algebra (James, 1957) whose idempotents are those required.
- It is possible to specify structure without a factor poset.

Idempotents for the RCBD

- Generalized factors and Sources





- Sets of idempotents (structures)
  - $\mathcal{U} = \{\mathbf{U}_{\text{Mean}}, \mathbf{U}_{\text{Rows}}, \mathbf{U}_{\text{Columns[Rows]}}\}$
  - $\mathcal{R} = \{\mathbf{R}_{\text{Mean}}, \mathbf{R}_{\text{A}}\}$
- Idempotents **U** and **R**, for each source, are a linear function of mean operators **M**:

$$\mathbf{M}_j = \mathbf{Z}_j\left(\mathbf{Z}'_j\mathbf{Z}_j\right)^{-1}\mathbf{Z}'_j \text{ or } \mathbf{M}_i = \mathbf{X}_i\left(\mathbf{X}'_i\mathbf{X}_i\right)^{-1}\mathbf{X}'_i$$

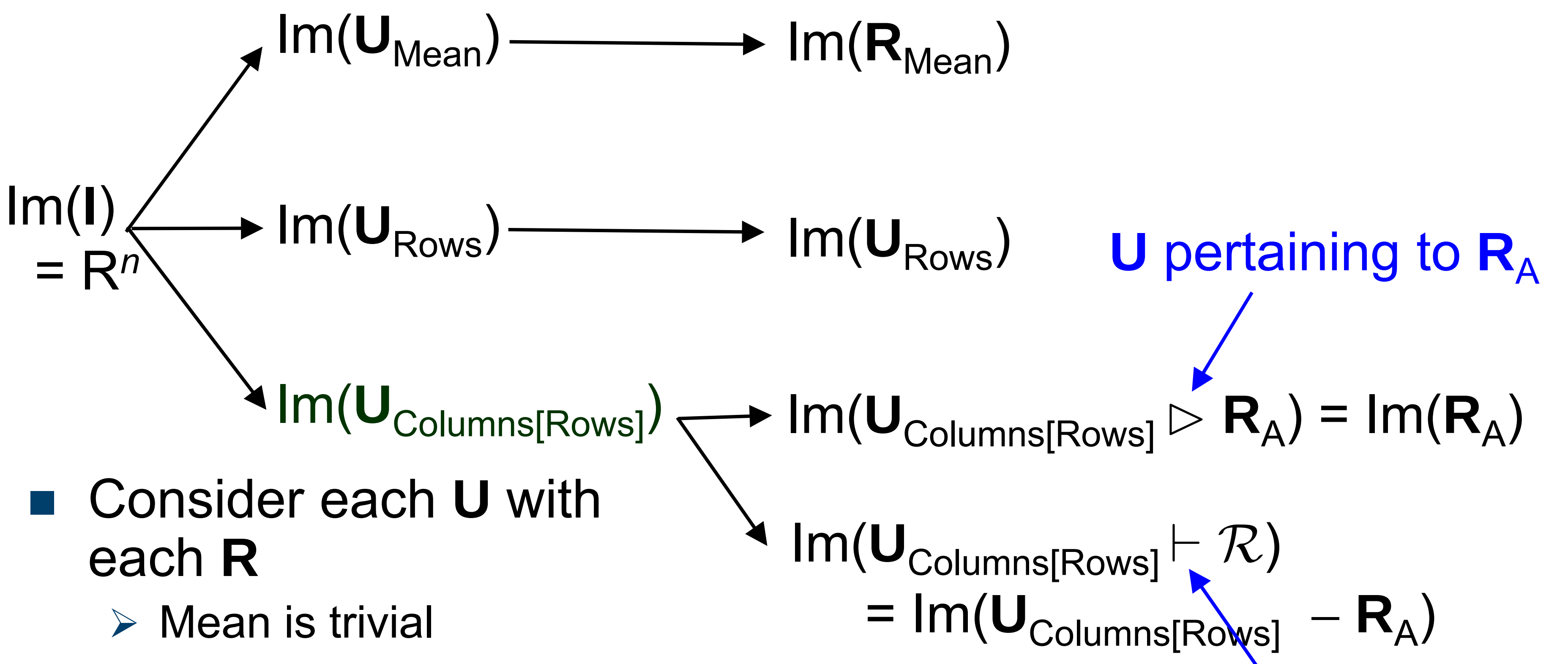
Expressions for **U**s and **R**s in terms of **M**s using Hasse diagrams

A pair of orthogonal decompositions

- Two orthogonal decompositions $\mathcal{U}$ and $\mathcal{R}$.
- The design is **orthogonal** if each subspace in $\mathcal{R}$ is contained in just one subspace in $\mathcal{U}$.
- That is, for each $\mathbf{R} \in \mathcal{R}$:
  - there is just one $\mathbf{U}_i \in \mathcal{U}$ such that $\mathbf{U}_i\mathbf{R} = \mathbf{R}\mathbf{U}_i = \mathbf{R}$;
  - for all other $\mathbf{U}_j \in \mathcal{U}$, $\mathbf{U}_j\mathbf{R} = \mathbf{R}\mathbf{U}_j = \mathbf{0}$.
- For our example, have
  - $\mathcal{U} = \{\mathbf{U}_{\text{Mean}}, \mathbf{U}_{\text{Rows}}, \mathbf{U}_{\text{Columns[Rows]}}\}$;
  - $\mathcal{R} = \{\mathbf{R}_{\text{Mean}}, \mathbf{R}_{\text{A}}\}$.
- So ignoring Mean sources, can show:
  - $\mathbf{U}_{\text{Rows}}\,\mathbf{R}_{\text{A}} = \mathbf{0} \Rightarrow$ Rows and A are orthogonal;
  - and $\mathbf{U}_{\text{Columns[Rows]}}\,\mathbf{R}_{\text{A}} = \mathbf{R}_{\text{A}}$
    $\Rightarrow$ A is totally confounded with Columns[Rows].

Decomposition: $\mathcal{U} \rhd \mathcal{R}$

$\mathcal{U} = \{\mathbf{U}_{\text{Mean}}, \mathbf{U}_{\text{Rows}}, \mathbf{U}_{\text{Columns[Rows]}}\} \rhd \mathcal{R} = \{\mathbf{R}_{\text{Mean}}, \mathbf{R}_{\text{A}}\}$



- Consider each **U** with each **R**
  - Mean is trivial
  - only need to consider $\mathbf{R}_{\text{A}}$.
- $\mathcal{U} \rhd \mathcal{R}$ is $\{\mathbf{U}_{\text{Mean}}, \mathbf{U}_{\text{Rows}}, \mathbf{U}_{\text{Columns[Rows]}} \rhd \mathbf{R}_{\text{A}}, \mathbf{U}_{\text{Columns[Rows]}} \vdash \mathcal{R}\}$ or $\{\mathbf{U}_{\text{Mean}}, \mathbf{U}_{\text{Rows}}, \mathbf{R}_{\text{A}}, (\mathbf{U}_{\text{Columns[Rows]}} - \mathbf{R}_{\text{A}})\}$.

Properties of design summarized in decomposition table

plots tier			treatments tier		
source	df	idempotent	source	df	idempotent
Rows	$b - 1$	$\mathbf{U}_{\text{Rows}}$			
Columns[Rows]	$b(a - 1)$	$\mathbf{U}_{\text{Columns[Rows]}}$	A	$a - 1$	$\mathbf{U}_{\text{Columns[Rows]}} \triangleright \mathbf{R}_{\text{A}} = \mathbf{R}_{\text{A}}$
			Residual	$(a - 1)(b - 1)$	$\mathbf{U}_{\text{Columns[Rows]}} \vdash \mathcal{R}$

- $\mathcal{U} \triangleright \mathcal{R}$ consists of 3 orthogonal subspaces (+ Mean)
- A further decomposes Columns[Rows] because A randomized to columns within rows
- Residual formed by difference
- If **Q** is an idempotent, then **y'Qy** is the SSq

Summary (so far)

- Two sets of objects, one randomized to other
- A tier is a set of factors indexing a set of objects, and hence determined by randomization
- Factor relationships within tiers determine generalized factors and sources for each tier
- One idempotent for each generalized factor/source
- Relationships between two sets of idempotents determine decomposition table
- Tiers are fundamental
- But so far just Nelder
  - unrandomized tier = block factors
  - randomized tier = treatment factors
- If only two sets of factors then do not need tiers, although I prefer unrandomized/randomized over block/treatment
  - Reflects key distinction and so less liable to misinterpretation.
  - Too many alternative meanings(nuisance/interest, random/fixed). Are Plots and Sex block factors?

What are properties of a standard split-plot experiment?

An experiment about cooking cakes in an oven

- *Main plots*
  - 4 Baking conditions randomized to 4 Intervals in each of 3 Days using RCBD.
- *Subplots*
  - In an Interval, an oven with 3 Shelves used to bake 3 different Recipes together.
  - Recipes order on Shelves randomized in each Interval.
- What are:
  - Sets of objects
  - Tiers
  - Factor posets and structure formulae
  - Generalized factors
  - Sources
  - Sets of idempotents
  - Idempotent relationships
  - Decomposition table

Structure-balanced experiments

- So orthogonal experiments are nice:
  - for every $\mathbf{R}\in\mathcal{R}$, $\mathbf{UR} = \mathbf{RU} = \mathbf{R}$ for one $\mathbf{U}\in\mathcal{U}$
- What other experiments are nice?
  - Balanced experiments
- Brien and Bailey (2008), following Nelder, call an experiment structure-balanced if, for all the matrices in structures $\mathcal{U}$ and $\mathcal{R}$, the following relations between the idempotents are met:
  - $\mathbf{RUR} = \lambda_{\mathbf{UR}}\mathbf{R}$ for all $\mathbf{R}\in\mathcal{R}$, $\mathbf{U}\in\mathcal{U}$
    ($0\leq\lambda_{\mathbf{UR}}\leq 1$ is called a canonical efficiency factor and is the nonzero eigenvalue of both **RUR** and **URU**.)
  - $\mathbf{R}_1\mathbf{UR}_2 = 0$ for all $\mathbf{U}\in\mathcal{U}$ and $\mathbf{R}_1 \neq \mathbf{R}_2$
    (implies that when orthogonal $\mathbf{R}_1$, $\mathbf{R}_2$ projected into Im(**U)** they remain orthogonal and so a unique analysis.)
  - Properties, w.r.t. efficiency, summarized in the $\mathcal{U} \times \mathcal{R}$ efficiency matrix $\Lambda_{\mathcal{UR}}$.

Relationship to reduced normal equations

- In block designs the canonical efficiency factors are the eigenvalues of $\mathbf{r}^{-\delta}\mathbf{A} = \mathbf{A}^*$,
  - where **A** is the information matrix from the reduced normal equations and $\mathbf{r}^{\delta}$ is the diagonal matrix of replications of the treatments.
- From James and Wilkinson (1971, Th. 3), in general, the eigenvalues of an **RUR** and its corresponding $\mathbf{A}^*$ are equal.
- That is, for a block design,
  - The eigenvalues of $\mathbf{R}_T\mathbf{U}_{P[B]}\mathbf{R}_T$ and $\mathbf{A}^*$ are the same.
  - Can be shown that, for a block design,

$$\mathbf{R}_T\mathbf{U}_{P[B]}\mathbf{R}_T = \mathbf{X}\mathbf{r}^{-(\delta/2)}\mathbf{A}^*\mathbf{r}^{-(\delta/2)}\,\mathbf{X}'$$

    - $\mathbf{R}_T$ is the orthogonal projector for treatments;
    - $\mathbf{R}_{P[B]}$ is the orthogonal projector for plots within blocks;
    - **X** is the design matrix for treatments.
- For a balanced design, the single eigenvalue is $\lambda_{\mathbf{UR}}$.

The decomposition

- Now define the idempotents that further refine $\mathcal{U}$ according to $\mathcal{R}$.

$$\mathbf{U} \triangleright \mathbf{R} = \lambda_{\mathbf{UR}}^{-1}\mathbf{URU} \qquad \mathbf{U} \text{ pertaining to } \mathbf{R}$$

$$\mathbf{U} \vdash \mathcal{R} = \mathbf{U} - \sum_{\mathbf{R}\in\mathcal{R}} \lambda_{\mathbf{UR}}^{-1}\mathbf{URU} \qquad \mathbf{U} \text{ orthogonal to all } \mathbf{R}$$

- Note that $\mathbf{R}_1\mathbf{U}\mathbf{R}_2 = 0$ for $\mathbf{R}_1 \neq \mathbf{R}_2$ implies $\mathbf{U}\triangleright\mathbf{R}_1$ and $\mathbf{U}\triangleright\mathbf{R}_2$ are orthogonal.
- So $\mathbf{U}\triangleright\mathbf{R}_1$, $\mathbf{U}\triangleright\mathbf{R}_2$, … $\mathbf{U}\triangleright\mathbf{R}_m$, $\mathbf{U}\vdash\mathcal{R}$ decompose **U** orthogonally.
- Then the complete decomposition is given by

$$\mathcal{U} \triangleright \mathcal{R} = \left\{\mathbf{U} \triangleright \mathbf{R} : \mathbf{U} \in \mathcal{U}, \mathbf{R} \in \mathcal{R}, \lambda_{\mathbf{UR}} \neq 0\right\} \cup \left\{\mathbf{U} \vdash \mathcal{R} : \mathbf{U} \in \mathcal{U}\right\}$$

- i.e. all bits of Im(**U**) with a bit of Im(**R**) in it and all bits of Im(**U**) without any Im(**R**) in it.

A BIBD — v = 6, b =15, k = 4, E = 0.9

d	**c**	f	e	f	e	d	a	f	a	b	d
c	f	e	a	e	a	b	c	d	e	a	b
e	a	**d**	**c**	d	f	e	b	c	f	b	a
c	e	**d**	b	f	b	**d**	**c**	**c**	a	**d**	f
f	e	b	c	a	e	f	b	a	**c**	b	**d**

- Each pair of treatments occurs together in exactly 6 blocks
- Sets of idempotents:
  - $\mathcal{U} = \{\mathbf{U}_{\text{Mean}}, \mathbf{U}_{\text{Blocks}}, \mathbf{U}_{\text{Plots[Blocks]}}\}$ and $\mathcal{R} = \{\mathbf{R}_{\text{Mean}}, \mathbf{R}_{\text{Treats}}\}$
- Relationships between sets:
  - $\mathbf{R}_{\text{Treats}}\, \mathbf{U}_{\text{Blocks}}\, \mathbf{R}_{\text{Treats}} = 1/10\ \mathbf{R}_{\text{Treats}}$
  - $\mathbf{R}_{\text{Treats}}\, \mathbf{U}_{\text{Plots[Blocks]}}\, \mathbf{R}_{\text{Treats}} = 9/10\ \mathbf{R}_{\text{treats}}$
- The efficiency matrix: $$\Lambda_{\mathcal{UR}} = \begin{array}{r} \\ \text{Mean} \\ \text{Blocks} \\ \text{Plots[B]} \end{array} \begin{array}{c} \begin{array}{cc} \text{Mean} & \text{Treat} \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \\ 0 & 0.9 \end{bmatrix} \end{array}$$
- The decomposition consists of 4 (+ Mean) elements:
  - $\mathcal{U} \rhd \mathcal{R} = \{\mathbf{U}_{\text{Mean}}, \mathbf{U}_{\text{B}} \rhd \mathbf{R}_{\text{T}}, \mathbf{U}_{\text{B}} \vdash \mathcal{R}, \mathbf{U}_{\text{P[B]}} \rhd \mathbf{R}_{\text{T}}, \mathbf{U}_{\text{P[B]}} \vdash \mathcal{R}\}$

Decomposition table for the BIBD

plots tier		treatments tier		
source	df	eff	source	df
Blocks	14	1/10	Treatments	5
			Residual	9
Plots[Blocks]	45	9/10	Treatments	5
			Residual	40

- Treatments partially confounded with both Blocks and Plot[Blocks] in proportion to efficiency factors:
  - — reflects $\mathbf{R}_{\text{Treats}}\,\mathbf{U}_{\text{Blocks}}\,\mathbf{R}_{\text{Treats}} = 1/10\;\mathbf{R}_{\text{Treats}}$ and $\mathbf{R}_{\text{Treats}}\,\mathbf{U}_{\text{Plots[Blocks]}}\,\mathbf{R}_{\text{Treats}} = 9/10\;\mathbf{R}_{\text{treats}}$
- This is a unique decomposition.
  - ➢ cf. Unadj Blocks + Adj Treats
    vs Unadj Treats + Adj Blocks

Software for decomposition (skeleton ANOVA) tables

- Useful to obtain for checking design.
- Several packages allow derivation of tables of the form described.
  - Use model formulae based on crossing and nesting.
  - Genstat (Payne et al., 2008) with `blockstructure` and `treatmentstructure`.
    - e.g. BIBD:
      ```
      block Blocks/Plots
      treat Treatments
      anova
      ```
  - R (R Core Development Team, 2008) and S-Plus (Insightful corporation, 2007) in `aov` function:
    - `Y ~ randomized formula + Error(unrandomized formula)`
    - e.g. BIBD: `Y ~ Treatments + Error(Blocks/Plots)`
    - Y can be randomly-generated data

Expected mean squares

- Often not crucial as clear what variability is affecting which sources
- If required, can use Hasse diagrams to assist in getting them.
  - Replace nos of levels with variance components premultiplied by replications of generalized factors or quadratic forms in the expectation.
  - Derive contributions for sources by adding variance components and their multipliers for generalized factors to which the source is marginal
- Add contributions to ANOVA table, multiplying by efficiencies.

Expected mean squares for RCBD

- Hasse diagrams (omitting Universe) are as follows:

Rows	$a\sigma^2_{\text{R}}$	$\boldsymbol{\sigma^2_{\text{RC}}} + a\sigma^2_{\text{R}}$	R	A $\quad q_{\text{A}}(\tau)$ • $q_{\text{A}}(\tau)$ $\quad$ A
Rows∧Columns	σ^2_{RC}	σ^2_{RC}	C[R]	

- Skeleton ANOVA table

plots tier		treatments tier		
source	df	source	df	E[MSq]
Mean	1	Mean	1	$\sigma^2_{\text{RC}} + a\sigma^2_{\text{R}}$
Rows	$b - 1$			
Columns[R]	$b(a - 1)$	A	$a - 1$	$\sigma^2_{\text{RC}} + q_{\text{A}}(\tau)$
		Residual	$(a - 1)(b - 1)$	σ^2_{RC}

- As was clear from the decomposition table, the Residual for Columns[Rows] provides the error for A.

3. Summary

- A randomization consists of two sets of objects, one randomized to other.
- A tier is a set of factors indexing a set of objects, and hence determined by randomization:
  - unrandomized tier = block factors;
  - randomized tier = treatment factors.
- Factor relationships within tiers determine generalized factors and sources for each tier.
- One idempotent for each generalized factor/source.
- Relationships between two sets of idempotents determine decomposition table that displays the confounding.
- Hasse diagrams useful for deriving degrees of freedom, idempotents in terms of mean operators and E[MSq]s.
- Multitiered experiments involve multiple randomizations.
- Need a tiered-based approach to sort out confounding in multitiered experiments.